

Exam Computational Methods of Science

June 18th 2025 11:45-13:45 (2 hours)

This exam is 'closed book'. It is NOT allowed to use a book, lecture notes or electronic devices.

Always give a clear explanation of your answer. An answer without any computation will not be rewarded, so also copy the computations from your scratch paper.

Free points: 10

Part 1

1. Consider the o.d.e. $U \frac{d\phi}{dx} = \lambda\phi + 6x$, with boundary conditions $\phi(0) = 0$, $\frac{d\phi}{dx}(1) = 0$.
The problem (i.e. both equation and b.c.) is discretised with 2nd order central discretisation (FDM) on an equidistantly spaced grid (gridsize h), where $x_0 = 0$, $x_N = 1$.
 - (a) 6 Derive the (local) truncation error, using Taylor series.
 - (b) 8 Describe the resulting linear system $A\vec{\phi} = \vec{b}$, for the solution vector $\vec{\phi} = [\phi_0 \cdots \phi_N]$.
 - (c) 6 (1) Derive for which U, λ, h the resulting matrix A is (weakly) diagonally dominant.
(2) Why is this relevant?
(3) Does grid refinement improve the situation? Explain why (not).
2. 10 Prove that for any discretisation (matrix) A_h : Coercivity \implies Analytic Stability.

Part 2

1. Consider for $x \in [0, 1]$ the p.d.e. $\frac{\partial\phi}{\partial t} + \underline{\alpha} \frac{\partial\phi}{\partial x} = D \frac{\partial^2\phi}{\partial x^2} + \underline{\beta} x \phi$, with $D = \frac{1}{40}$.
The boundary and initial conditions are $\phi(0, t) = 0$, $\phi(1, t) = 1$, $\phi(x, 0) = x^2$.
The equidistant spatial grid consists of 9 internal gridpoints and 2 boundary points.
For $\frac{\partial^2\phi}{\partial x^2}$ central FDM discretisation is used.
 - (a) 4 Let $\alpha = \beta = 0$ (so the underlined terms are 'switched off').
 - (1) What is the max. possible time step δt in case of the Explicit Euler method?
 - (2) Does this limit change, if the initial profile is given by $\phi(x, 0) = x$? Explain.
 - (b) 6 Let $\alpha = 0$, $\beta = 100$. Determine the value of $\phi(\frac{1}{5}, \frac{1}{10})$, using Explicit Euler with $\delta t = \frac{1}{10}$.
Hint: set up the discretisation first.
 - (c) 10 Let $\alpha = 1$, $\beta = 100$ (so all underlined terms become 'active').
The extra term $\frac{\partial\phi}{\partial x}$ is treated with 1st order backward(!) FDM discretisation.
Determine the row of the linear system (i.e. values of the entries of the matrix and RHS vector) that is related to $x = \frac{1}{5}$, when Implicit Euler is used with $\delta t = \frac{1}{10}$.
2. 10 The p.d.e. $\frac{\partial\phi}{\partial t} = D \frac{\partial^2\phi}{\partial x^2}$ is treated with 2nd order central spatial discretisation (FDM), on an equidistantly spaced grid, in combination with the Explicit Euler method.
Apply Von Neumann stability analysis, to determine the (practical) stability limit.
Hint: use $\phi_j^{(n)} = \rho^n e^{i j \varphi}$ in the discretisation and determine the amplification factor.

P.T.O.

Part 3

1. Consider for $x \in [0, 2]$ the o.d.e. $-\frac{d^2u}{dx^2} + 2u = 3x$, with bound. cond. $u(0) = \frac{du}{dx}(2) = 0$.

- (a) **10** Using the Galerkin (weighted-residual) approach, determine the weak formulation of the problem and identify the bi-linear and linear forms.
- (b) **10** (1) Show that the bi-linear form is positive definite.
(2) Show that the linear form is bounded.
- (c) **10** The domain will be covered with 3 grid points: $x_0 = 0, x_1 = 1, x_2 = 2$.
We will use piecewise linear ('tent shape') basis functions, with $\phi_i(x_j) = \delta_{ij}$.
Use integration to evaluate $A_{1,1}$ and b_1 , which are part of the middle row of the linear system $A\vec{c} = \vec{b}$, where $\vec{c} = [c_0 \cdots c_2]$.

Total: **100**

Write your name and student number on each page!